

You must show all work!

1. Find the **general solution** of the differential equation.

$$\frac{dy}{dx} = 5x^4 - 9x^3$$

$$dy = (5x^4 - 9x^3) dx$$

$$\int 1 dy = \int (5x^4 - 9x^3) dx$$

$$y = \frac{5x^5}{5} - \frac{9x^4}{4} + C$$

$$y = x^5 - \frac{9}{4}x^4 + C$$

2. Find the **general solution** of the differential equation.

$$y' = \frac{4x}{y}$$

$$y dy = 4x dx \quad \text{Separate Variables}$$

$$\int y dy = \int 4x dx$$

$$\frac{1}{2}y^2 = \frac{4}{2}x^2 + C_1$$

$$\frac{1}{2}y^2 = 2x^2 + C_1$$

$$2\left(\frac{1}{2}y^2\right) = 2(2x^2 + C_1)$$

$$y^2 = 4x^2 + 2C_1 \quad \text{Let } C = 2C_1$$

$$y^2 = 4x^2 + C$$

3. Find the **general solution** to the differential equation. Then find the **particular solution** with initial condition $y(1) = 5$.

$$y' - 3x^2y = 0$$

$$\frac{dy}{dx} = 3x^2y$$

$$\frac{1}{y} \frac{dy}{dx} dx = \frac{3x^2y}{y} dx$$

$$\frac{1}{y} dx = 3x^2 dx \quad \text{Separate Variables}$$

$$\int \frac{1}{y} dx = \int 3x^2 dx \quad \text{Now, integrate Both sides.}$$

$$\ln|y| = \frac{3x^3}{3} + C_1$$

$$\ln|y| = x^3 + C_1$$

$$\ln|y| = x^3 + C_1$$

$$e^{\ln|y|} = e^{x^3+C_1} = e^{x^3} e^{C_1} \text{ let } C = e^{C_1}$$

$$|y| = Ce^{x^3} \quad \text{General Solution}$$

Plug in (1,5)

$$|5| = Ce^{1^3} = ce$$

$$5 = Ce \Rightarrow C = \frac{5}{e}$$

$$y = \frac{5}{e} e^{x^3} = 5e^{-1} e^{x^3} = 5e^{x^3-1}$$

$$y = 5e^{x^3-1} \quad \text{Particular Solutions}$$

We could try using the first order technique. It works either way much to my surprise!

$$y' - 3x^2y = 0$$

$$y' + P(x)y = Q(x) \text{ so } P(x) = -3x^2 \text{ and } Q(x) = 0$$

$$u(x) = e^{\int -3x^2 dx} = e^{-x^3}$$

$$y = \frac{1}{u(x)} \int Q(x)u(x) dx = \frac{1}{e^{-x^3}} \int 0(e^{-x^3}) dx = \frac{1}{e^{-x^3}} (0 + C)$$

$$y = Ce^{x^3} \quad \text{General Solution}$$

plug in (1,5)

$$5 = Ce^{1^3} = Ce \Rightarrow C = \frac{5}{e}$$

$$y = \frac{5}{e} e^{x^3} = 5e^{-1} e^{x^3} = 5e^{x^3-1}$$

$$y = 5e^{x^3-1} \quad \text{Particular Solution}$$

4. Find the general solution of the differential equation $y' + 2xy = 10x$.

$$y' + P(x)y = Q(x) \text{ so } P(x) = 2x \text{ and } Q(x) = 10x$$

$$u(x) = e^{\int 2x dx} = e^{x^2}$$

$$y = \frac{1}{u(x)} \int Q(x)u(x) dx = \frac{1}{e^{x^2}} \int 10xe^{x^2} dx = \frac{10}{e^{x^2}} \int xe^{x^2} dx$$

$$y = \frac{10}{e^{x^2}} \int e^u du = \frac{10}{e^{x^2}} \left(\frac{1}{2} e^{x^2} + C \right) = 5 + Ce^{-x^2}$$

$$y = 5 + Ce^{-x^2} \quad \text{General Solution}$$

$$\int xe^{x^2} dx = \frac{1}{2} \int e^u du = \frac{1}{2} e^u + C = \frac{1}{2} e^{x^2} + C$$

$$\text{Let } u = x^2, du = 2x dx \Rightarrow \frac{1}{2} du = x dx$$

You can separate this too. $\frac{dy}{dx} = 10x - 2xy = 2x(5 - y) \Rightarrow \frac{1}{5-y} dy = 2x dx \text{ or } \frac{1}{y-5} dy = -2x dx$

5. Find the general solution of the differential equation $y' + 6y = e^{6x}$.

$$y' + P(x)y = Q(x) \text{ so } P(x) = 6 \text{ and } Q(x) = e^{6x}$$

$$u(x) = e^{\int 6 dx} = e^{6x}$$

$$y = \frac{1}{u(x)} \int Q(x)u(x) dx = \frac{1}{e^{6x}} \int e^{6x} e^{6x} dx = \frac{1}{e^{6x}} \int e^{12x} dx$$

$$y = \frac{1}{e^{6x}} \int \frac{1}{12} e^u du = \frac{1}{e^{6x}} \left(\frac{1}{12} e^{12x} + C \right) = \frac{1}{12} e^{6x} + C e^{-6x}$$

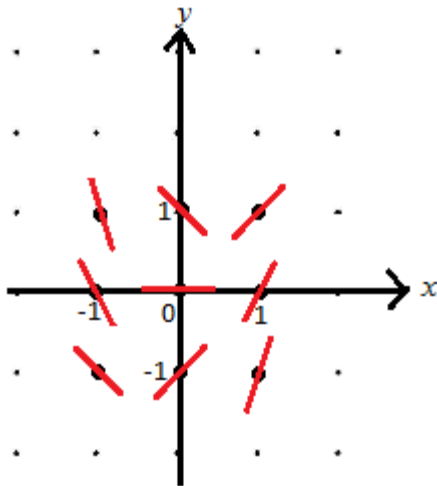
let $u=12x, du=12 dx$

$$y = \frac{1}{12} e^{6x} + C e^{-6x} \text{ General Solution}$$

6. Sketch the **slope field** for the differential equation. $y' = 2x - y$

Use the nine points that are bold and larger.

Can this be separated? Explain?



x	y	$y' = 2x - y$
-1	-1	$-2 + 1 = -1$
-1	0	$-2 + 0 = -2$
-1	1	$-2 - 1 = -3$
0	-1	$0 + 1 = 1$
0	0	$0 - 0 = 0$
0	1	$0 - 1 = -1$
1	-1	$2 + 1 = 3$
1	0	$2 - 0 = 2$
1	1	$2 - 1 = 1$

7. Write the differential equation for the following statement.

The rate of change of the variable y is proportional to the value of y .
Let y be a function of time t .

Show that the solutions are in the form of $y = Ce^{kt}$

$$\begin{array}{ll}
 y' = ky & \\
 \frac{dy}{dt} = ky & e^{\ln|y|} = e^{kt+C_1} \\
 \frac{1}{y} dy = k dt & e^{\ln|y|} = e^{kt} e^{C_1} \quad \text{let } C = e^{C_1} \\
 \int \frac{1}{y} dy = \int k dt & |y| = Ce^{kt} \quad \text{if } y > 0, \text{ then} \\
 & y = Ce^{kt}
 \end{array}$$

Note: We need a new constant. $C \neq e^C$ There is no value of C that makes this true?

$$5 \neq e^5 \text{ or } 3 \neq e^3 \text{ we let get a new constant, } C = e^{C_1}$$

8. Assume that $y = Ce^{kt}$ is a general solution to a Radioactive Decay differential equation with the initial conditions: $y = 50$ when $t = 0$. Let y represent the mass in grams of carbon. The half-life of carbon isotopes is 5715 years.

- Find the particular equation and
- then find how long it would take for 50 grams to decay to 10 grams.

$$y = Ce^{kt} \text{ If } t = 0, \text{ when } y = 50.$$

$$50 = Ce^{k(0)} = Ce^0 = C(1) = C, \text{ so } C = 50.$$

$$y = 50e^{kt}. \text{ Now find } k.$$

$$y = 50e^{kt} \text{ there is 25 grams left after 5715 years. } y = 50e^{kt} \text{ where } k = \frac{\ln\left(\frac{1}{2}\right)}{5715}.$$

$$25 = 50e^{k(5715)} \Rightarrow \frac{1}{2} = e^{5715k}$$

$$\ln\left(\frac{1}{2}\right) = \ln e^{5715k} \Rightarrow 5715k = \ln\left(\frac{1}{2}\right)$$

$$k = \frac{\ln\left(\frac{1}{2}\right)}{5715}$$

$$10 = 50e^{kt}$$

$$\frac{1}{5} = e^{kt} \Rightarrow \ln\left(\frac{1}{5}\right) = kt$$

$$t = \frac{\ln\left(\frac{1}{5}\right)}{\frac{\ln\left(\frac{1}{2}\right)}{5715}} = \frac{5715 \ln\left(\frac{1}{5}\right)}{\ln\left(\frac{1}{2}\right)} \approx 13,269.82 \text{ years}$$

9. Use Euler's method to approximate the particular solution to $\frac{dy}{dx} = x + y$ passing through the point (0,1). Use 4 steps of 0.5.

x	$y_n = y_{n-1} + hF(x_{n-1}, y_{n-1})$
$x_0 = 0$	$y_0 = 1$
$x_1 = 0.5$	$y_1 = 1 + .5(0+1) = 1.5$
$x_2 = 1$	$y_2 = 1.5 + .5(.5+1.5) = 2.5$
$x_3 = 1.5$	$y_3 = 2.5 + .5(1+2.5) = 4.25$
$x_4 = 2$	$y_4 = 4.25 + .5(1.5+4.25) = 7.125$

10. Find the general solution of the differential equation $\frac{dy}{dx} = 5x$.

$$dy = 5x dx$$

$$\int dy = \int 5x dx$$

$$y = \frac{5}{2}x^2 + C$$