

McDonald

*You must show all work! 6 points each***Evaluate the integral. Use integration by parts**

1. $\int x e^{-3x} dx$

$$\int x e^{-3x} dx = uv - \int v du = x \left(-\frac{1}{3} e^{-3x} \right) - \int -\frac{1}{3} e^{-3x} dx$$

$$= \frac{1}{3} x e^{-3x} + \frac{1}{3} \int e^{-3x} dx = \frac{1}{3} x e^{-3x} + \frac{1}{3} \left(-\frac{1}{3} e^{-3x} \right) + C$$

$$= \frac{1}{3} x e^{-3x} - \frac{1}{9} e^{-3x} + C$$

$u = x \quad dv = e^{-3x} dx$

$du = dx \quad v = -\frac{1}{3} e^{-3x}$

Just set up using partial fractions. Do not evaluate. Ask if you are confused.

2. $\int \frac{3x^2 + 4x - 6}{(x-3)^2(x^2+9)} dx$

Set up for partial fraction.

$$\frac{3x^2 + 4x - 6}{(x-3)^2(x^2+9)} = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{Cx+D}{x^2+9}$$

Do NOT do this!!

$$\int \frac{3x^2 + 4x - 6}{(x-3)^2(x^2+9)} dx = \frac{A}{x-3} + \frac{B}{(x-3)^2} + \frac{Cx+D}{x^2+9} \quad \text{Not equal!}$$

The LHS side is the integral. We haven't taken the integral of the RHS yet.

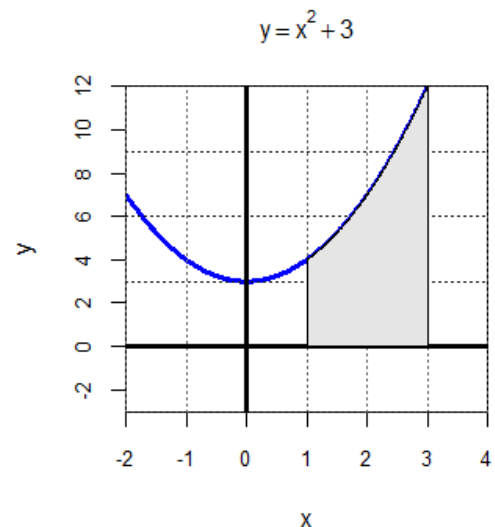
3. Find the **area** of the region bounded by the graphs: Please Graph!

$y = x^2 + 3, y = 0, x = 1 \text{ and } x = 3$

$$A = \int_1^3 (x^2 + 3 - 0) dx = \frac{1}{3} x^3 + 3x \Big|_1^3 = \frac{1}{3} 3^3 + 3(3) - \left(\frac{1}{3} 1^3 + 3(1) \right)$$

$$= 9 + 9 - \frac{1}{3} - 3 = 15 - \frac{1}{3} = \frac{45-1}{3} = \frac{44}{3}$$

$$\frac{44}{3} \text{ units}^2$$



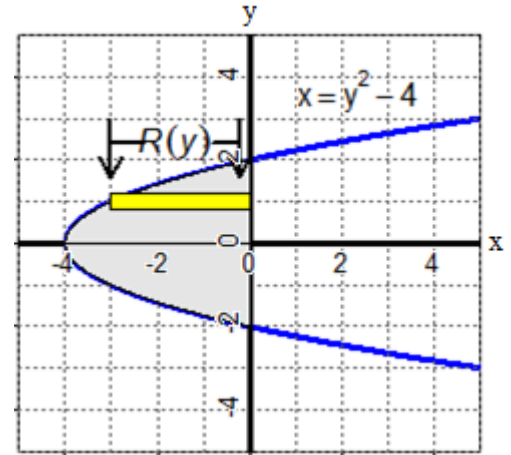
4. Find the **volume** of the solid formed by revolving the region bounded by $x = y^2 - 4, x = 0$ about the line y -axis. Use **Any Method**.

$$V = \frac{512\pi}{15} \text{ units}^3$$

The Washer (or Disk) method

$$R(y) = y^2 - 4, \quad r(y) = 0$$

$$\begin{aligned} V &= \pi \int_{-2}^2 (y^2 - 4)^2 dy = \pi \int_{-2}^2 (y^4 - 8y^2 + 16) dy \\ &= \pi \left[\frac{1}{5}y^5 - \frac{8}{3}y^3 + 16y \right]_{-2}^2 \\ &= \pi \left[\left(\frac{1}{5}(2)^5 - \frac{8}{3}(2)^3 + 16(2) \right) - \left(\frac{1}{5}(-2)^5 - \frac{8}{3}(-2)^3 + 16(-2) \right) \right] \\ &= \pi \left[\frac{256}{15} - \left(-\frac{256}{15} \right) \right] = \pi \left[\frac{256}{15} + \frac{256}{15} \right] = \frac{512\pi}{15} \end{aligned}$$

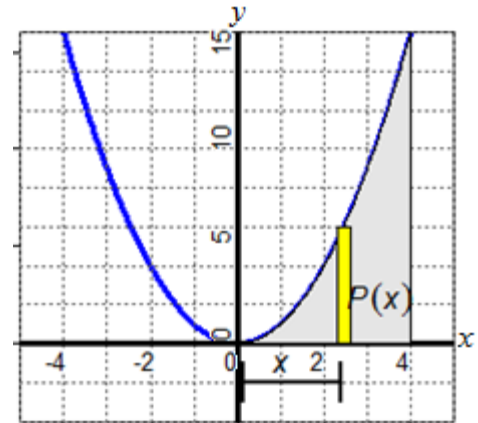


5. - 6. Find the **volume** of the solid formed by revolving the region bounded by $y = x^2, y = 0,$ and $x = 4$

5. about the line y -axis. Use **Shell Method**.

$$V = 2\pi \int_0^4 x \cdot x^2 dx = 2\pi \frac{1}{4} x^4 \Big|_0^4 = \frac{\pi}{2} 4^4 = 128\pi$$

$$V = 128\pi \text{ unit}^3$$

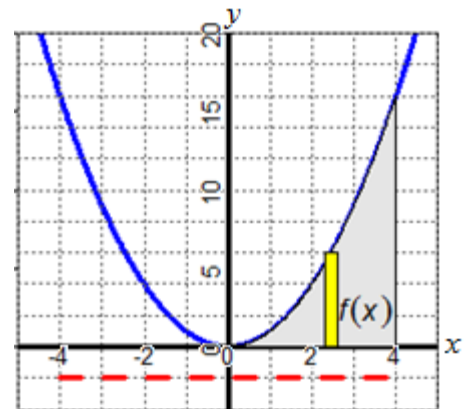


6. about the line $y = -2$. Use **Disk Method**.

$$V = \frac{4352\pi}{15} \text{ unit}^3$$

$$R(x) = f(x) - (-2) = x^2 + 2; \quad r(x) = 0 - (-2) = 2$$

$$\begin{aligned} V &= \pi \int_0^4 [(x^2 + 2)^2 - (2)^2] dx = \pi \int_0^4 [x^4 + 4x^2 + 4 - 4] dx \\ &= \pi \left[\frac{1}{5}x^5 + \frac{4}{3}x^3 \right]_0^4 = \pi \left[\frac{4^5}{5} + \frac{4^3}{3} - 0 \right] = \frac{4352\pi}{15} \end{aligned}$$



Evaluate the integral.

$$\begin{aligned} \text{Let } x &= 2 \sin \theta & \sqrt{4-x^2} &= \sqrt{4-(2 \sin \theta)^2} = \sqrt{4-4 \sin^2 \theta} = \sqrt{4(1-\sin^2 \theta)} \\ & & &= \sqrt{4 \cos^2 \theta} = 2 \cos \theta \\ 7. \quad \int \sqrt{4-x^2} dx & & dx &= 2 \cos \theta d\theta \\ & & \sin \theta &= \frac{x}{2} \end{aligned}$$

$$\int \sqrt{4-x^2} dx = \int (2 \cos \theta)(2 \cos \theta) d\theta = \int 4 \cos^2 \theta d\theta$$

$$\text{Use } \cos^2 \theta = \frac{1 + \cos(2\theta)}{2}.$$

$$\begin{aligned} 4 \int \cos^2 \theta d\theta &= 4 \int \frac{1 + \cos(2\theta)}{2} d\theta = 2 \int (1 + \cos(2\theta)) d\theta = 2 \left(\theta + \frac{1}{2} \sin(2\theta) \right) + C \\ &= 2\theta + \sin(2\theta) + C = 2\theta + 2 \sin \theta \cos \theta + C \\ &= 2 \sin^{-1} \left(\frac{x}{2} \right) + (2) \frac{x}{2} \frac{\sqrt{4-x^2}}{2} + C \\ &= 2 \sin^{-1} \left(\frac{x}{2} \right) + \frac{x\sqrt{4-x^2}}{2} + C \end{aligned}$$

Evaluate the integral.

$$\begin{aligned} 8. \quad \int \sin^3 x \cos^2 x dx & & \text{Let } u &= \cos x \\ & & du &= -\sin x dx \end{aligned}$$

$$\begin{aligned} \int \sin^3 x \cos^2 x dx &= \int \sin x \sin^2 x \cos^2 x dx = \int (1 - \cos^2 x) \cos^2 x \sin x dx \\ &= -\int (1 - u^2) u^2 du = -\int (u^2 - u^4) du = \int (u^4 - u^2) du \\ &= \frac{1}{5} u^5 - \frac{1}{3} u^3 + C \\ &= \frac{1}{5} \cos^5 x - \frac{1}{3} \cos^3 x + C \end{aligned}$$

Evaluate the following integrals

9. $\int \sin^2 x \, dx$

Just memorize these two identities: $\sin^2 \theta = \frac{1 - \cos(2\theta)}{2}$ and $\cos^2 \theta = \frac{1 + \cos(2\theta)}{2}$

$$\int \sin^2 x \, dx = \int \frac{1 - \cos(2x)}{2} \, dx = \int \left(\frac{1}{2} - \frac{1}{2} \cos(2x) \right) dx = \frac{1}{2}x - \frac{1}{2} \cdot \frac{1}{2} \sin(2x) + C$$

$$= \frac{1}{2}x - \frac{1}{4} \sin(2x) + C$$

10. $\int \ln x \, dx$ Hint: Use integration by parts.

Let $u = \ln x$ $dv = dx$

$du = \frac{1}{x} dx$ $v = x$

$$\int \ln x \, dx = uv - \int v du = (\ln x)x - \int x \left(\frac{1}{x} \right) dx = x \ln x - \int 1 \, dx$$

$$= x \ln x - x + C$$

Note: There is no reason to factor $x(\ln x - 1) + C$. Factoring is the opposite of simplifying.

Evaluate the following integrals

11. $\int \frac{1}{x^2 - 4x + 8} \, dx$

Looks like arctan x ?

Let $u = x - 2$ $a = 2$

$du = dx$

$$\int \frac{1}{x^2 - 4x + 8} \, dx = \int \frac{1}{x^2 - 4x + 4 + 4} \, dx = \int \frac{1}{(x-2)^2 + 4} \, dx = \int \frac{1}{(x-2)^2 + 2^2} \, dx$$

$$= \int \frac{1}{u^2 + 2^2} \, du = \frac{1}{2} \tan^{-1} \left(\frac{x-2}{2} \right) + C$$

$$12. \int \frac{x+4}{x^2-4x} dx$$

Use partial fractions.

$$\frac{x+4}{x(x-4)} = \frac{A}{x} + \frac{B}{x-4}$$

$$\left[\frac{x+4}{x(x-4)} \right] (x(x-4)) = \left[\frac{A}{x} + \frac{B}{x-4} \right] (x(x-4))$$

$$x+4 = A(x-4) + Bx$$

$$\text{Let } x = 0,$$

$$\text{Let } x = 4,$$

$$0+4 = A(0-4) + B(0) \quad 4+4 = A(4-4) + B(4)$$

$$4 = -4A$$

$$8 = 4B$$

$$A = -1$$

$$B = 2$$

$$\int \frac{x+4}{x^2-4x} dx = \int \left(\frac{-1}{x} + \frac{2}{x-4} \right) dx = -\ln|x| + 2\ln|x-4| + C = \ln \left| \frac{(x-4)^2}{x} \right| + C = \ln \frac{(x-4)^2}{|x|} + C$$

Anyone of these three forms is correct.

Evaluate the integral. Memorized answer or prove if you must.

$$13. \int \frac{2}{x\sqrt{x^2-4}} dx = \int \frac{2}{x\sqrt{x^2-2^2}} dx = 2 \left(\frac{1}{2} \sec^{-1} \left(\frac{|x|}{2} \right) \right) + C = \sec^{-1} \left(\frac{|x|}{2} \right) + C \quad \text{Done!!}$$

Bonus Points for using trig substitution. State trig substitution.

13. Continue... Bonus Points for using trig substitution. State trig substitution.

$$\int \frac{2}{x\sqrt{x^2-4}} dx = \quad \begin{array}{ll} x = 2 \sec \theta & \sec \theta = \frac{x}{2} \\ dx = 2 \sec \theta \tan \theta d\theta & \theta = \sec^{-1} \left(\frac{x}{2} \right) \end{array}$$

$$\sqrt{(2 \sec \theta)^2 - 4} = \sqrt{4 \sec^2 - 4} = \sqrt{4(\sec^2 \theta - 1)} = \sqrt{4 \tan^2 \theta} = 2 \tan \theta$$

$$\int \frac{2}{x\sqrt{x^2-4}} dx = \int \frac{2 \cdot 2 \sec \theta \tan \theta}{2 \sec \theta \cdot 2 \tan \theta} d\theta = \int 1 d\theta = \theta + C = \sec^{-1} \left(\frac{x}{2} \right) + C ; x > 2$$

This is a pretty cool problem!

14. Evaluate the integral. $\int x^4 \cos(2x) dx$

Hint: Use the table or chart method.

<i>sign</i>	u'	$\int dv$
+	x^4	$\cos(2x)$
-	$4x^3$	$\frac{1}{2} \sin(2x)$
+	$12x^2$	$-\frac{1}{4} \cos(2x)$
-	$24x$	$-\frac{1}{8} \sin(2x)$
+	24	$\frac{1}{16} \cos(2x)$
-	0	$\frac{1}{32} \sin(2x)$

<i>sign</i>	u'	$\int dv$
+	x^4	$\cos(2x)$
-	$4x^3$	$\frac{1}{2} \sin(2x)$
+	$12x^2$	$-\frac{1}{4} \cos(2x)$
-	$24x$	$-\frac{1}{8} \sin(2x)$
+	24	$\frac{1}{16} \cos(2x)$
-	0	$\frac{1}{32} \sin(2x)$

$$\int x^4 \cos(2x) dx = \frac{1}{2} x^4 \sin(2x) + x^3 \cos(2x) - \frac{3}{2} x^2 \sin(2x) - \frac{3}{2} x \cos(2x) + \frac{2}{3} \sin(2x) + C$$

Extra credit.

Solve one of the following: (circle the one you choose)

6 points $\int \sec^3 x dx$ or

5 points $\int \sin^{-1} x dx$

Both in book