

**Calculus Test 2**

**Name:** \_\_\_\_\_

*You must show all work!*

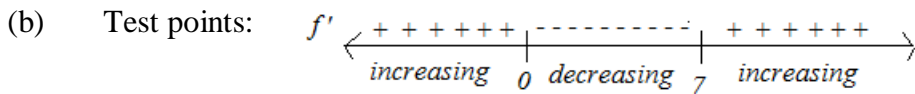
*7 points each*

**Date:** 13 November 2008

1.  $f(x) = 2x^3 - 21x^2 + 5$  Don't Graph!

- (a) Find all critical numbers for the function.
- (b) Find the intervals where the function is increasing or decreasing.
- (c) Apply the first or second derivative to find the extrema.

(a)  $f'(x) = 6x^2 - 42x = 6x(x - 7) = 0; c = 0, 7$



$$f'(-1) = 6(-1)(-1 - 7) = -6(-8) = 48 > 0$$

$$f'(11) = 6(11)(11 - 7) = 6(44) = 264 > 0$$

$$f'(8) = 6(8)(8 - 7) = 6(8) = 48 > 0$$

(c) First derivative Test: (FDT)

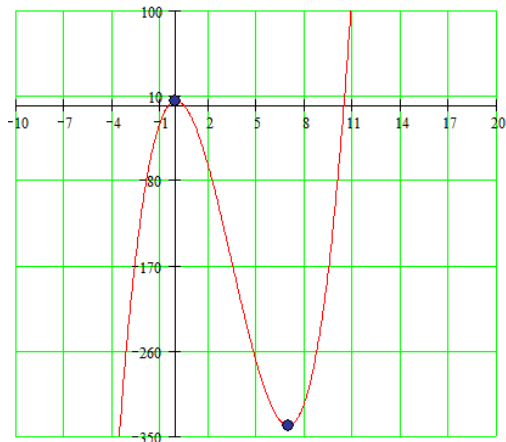
Since  $f'(x) > 0$  on  $(-\infty, 0)$  and  $f'(x) < 0$  on  $(0, 7)$ , by the FDT,  $(0, 5)$  is a relative maximum.

Since  $f'(x) < 0$  on  $(0, 7)$  and  $f'(x) > 0$  on  $(7, \infty)$ , by the FDT,  $(7, -338)$  is a relative minimum.

Remark:

$$f(0) = 0 + 0 + 5 = 5$$

$$f(7) = 2(7^3) - 21(7^2) + 5 = -338$$



You could also use the second derivative test.

$$f''(x) = 12x - 42$$

$$f''(0) = 12(0) - 42 = -42 < 0, \therefore CD, (0, 5) \text{ is rel. max.}$$

$$f''(7) = 12(7) - 42 = 42 > 0, \therefore CU, (7, -338) \text{ is rel. min.}$$

2.  $f(x) = \frac{x^2}{x^2 + 4}$       Don't Graph!

- (a) Find all critical numbers for the function.
- (b) Find the intervals where the function is increasing or decreasing.
- (c) Apply the first or second derivative to find the extrema.

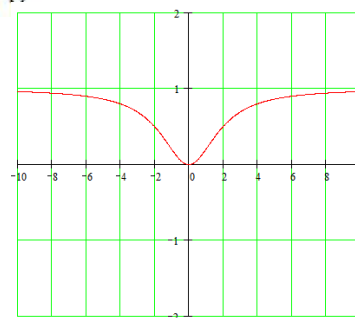
(a)  $f'(x) = \frac{(x^2 + 4)2x - 2x(x^2)}{(x^2 + 4)^2} = \frac{2x^3 + 8x - 2x^3}{(x^2 + 4)^2} = \frac{8x}{(x^2 + 4)^2}$

$c = 0$ , **note:**  $x^2 + 4 \neq 0$

(b) Test points:  $f'$   $\leftarrow$  decreasing  $0$  increasing  $\rightarrow$

$$f'(-1) = \frac{8(-1)}{(-1)^2 + 4} = \frac{-8}{5} < 0$$

$$f'(1) = \frac{8(1)}{(1)^2 + 4} = \frac{8}{5} > 0$$



(c) First derivative Test: (FDT)

Since  $f'(x) < 0$  on  $(-\infty, 0)$  and  $f'(x) > 0$  on  $(0, \infty)$ , by the FDT,  $(0, 0)$  is a relative minimum.

Remark:  $f(0) = 0$

You can use the Second Derivative Test but it is harder.

$$f''(x) = \frac{(x^2 + 4)^2 \cdot 8 - 8x(2)(x^2 + 4)(2x)}{[(x^2 + 4)^2]^2} = \frac{8(x^2 + 4)[(x^2 + 4) - x(2)(2x)]}{(x^2 + 4)^4}$$

$$= \frac{8(x^2 + 4)[x^2 + 4 - 4x^2]}{(x^2 + 4)^4} = \frac{8(4 - 3x^2)}{(x^2 + 4)^3}$$

$$f''(0) = \frac{8(4 - 3(0^2))}{(0^2 + 4)^3} = \frac{32}{64} > 0, \therefore CU. \text{ Hence } (0, 0) \text{ is a rel. min..}$$

3. Graph:  $g(x) = x^3 - 6x^2 + 6$ .  
Find all important information. (maxs, mins, poi, etc...) Skip  $x$ -intercepts.

$$g'(x) = 3x^2 - 12x = 3x(x - 4), \quad c = 0, 4$$

$$g''(x) = 6x - 12 = 6(x - 2), \quad k = 2$$

Using 2<sup>nd</sup> derivative test,

$$g''(0) = 6(0) - 12 = -12 < 0, \therefore CD, (0, 6) \text{ is a rel. max.}$$

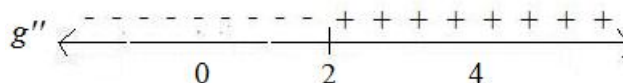
$$g''(4) = 6(4) - 12 = 12 > 0, \therefore CU, (4, -26) \text{ is a rel. min.}$$

Checking for POI,  $k = 2$ .

From above,

$$g''(0) < 0 \Rightarrow \text{Concave Down}$$

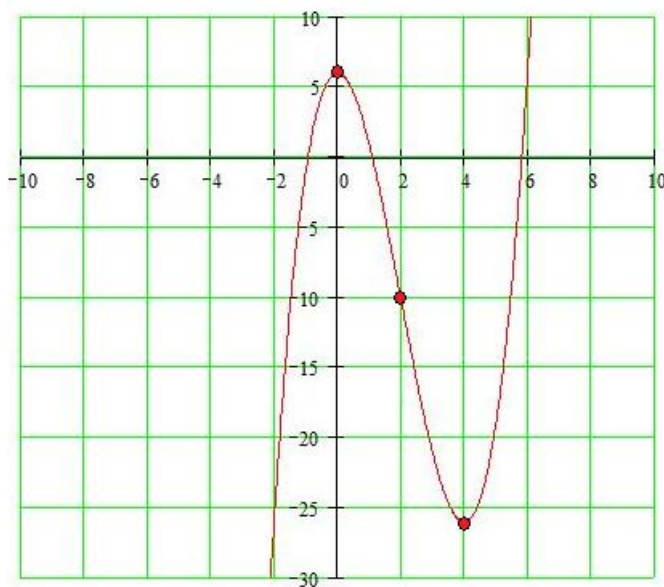
$$g''(4) > 0 \Rightarrow \text{Concave Up}$$



POI:  $(2, -10)$

$$\text{Remark: } g(2) = 2^3 - 6(2^2) + 6 = 8 - 24 + 6 = -10$$

Of course the  $y$ -intercept is  $(0, 6)$ .



4. **Answer and explain:**

- (a) If  $f'(3) = 0$  and  $f''(3) = -5$ , then is  $f(3)$  a relative maximum, minimum, p.o.i., or neither.

By the 2<sup>nd</sup> derivative test,  $f(3)$  is a relative maximum because 3 is a critical number and  $f''(3) < 0$  means that  $f$  is Concave Down.

- (b) If  $f'(3) = 0$  and  $f''(3) = 0$ , then is  $f(3)$  a relative maximum, minimum, p.o.i., or neither.

It is inconclusive if  $f(3)$  is a max or min by the second derivative test since  $f''(3) = 0$  is neither positive or negative. But, since  $f''(3) = 0$  is a **possible** Point Of Inflection. WE cannot tel for sure unless we know the concavity of the intervals on both sides of 3.

5. 
$$\lim_{n \rightarrow \infty} \sum_{i=1}^n \frac{i-3}{n^2} = \lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{i=1}^n (i-3) \quad (\text{notice hint})$$

$$\begin{aligned} \lim_{n \rightarrow \infty} \frac{1}{n^2} \sum_{i=1}^n (i-3) &= \lim_{n \rightarrow \infty} \frac{1}{n^2} \left[ \frac{n(n+1)}{2} - 3n \right] = \lim_{n \rightarrow \infty} \left[ \frac{n(n+1)}{2n^2} - \frac{3n}{n^2} \right] \\ &= \lim_{n \rightarrow \infty} \left[ \frac{n^2 + n}{2n^2} - \frac{3}{n} \right] = \frac{1}{2} - 0 = \frac{1}{2} \end{aligned}$$

6. Investigate the concavity of the following function. Must show; graphing doesn't count!

$$f(x) = \sqrt{x}$$

$$f(x) = x^{1/2}, x \geq 0$$

$$f'(x) = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}}; x > 0$$

$$f''(x) = -\frac{1}{2} \cdot \frac{1}{2}x^{-1/2-1} = -\frac{1}{4}x^{-3/2} = -\frac{1}{4\sqrt[2]{x^3}} = -\frac{1}{4x\sqrt{x}}; x \neq 0$$

We only need to test  $x > 0$ .

Let  $x = 1$ ,

$$f''(1) = -\frac{1}{4(1)\sqrt[2]{1}} = -\frac{1}{4} < 0$$

Therefore,  $f$  is concave down on the entire domain of  $[0, \infty)$ .

7. Find the horizontal asymptotes for  $f(x) = \frac{3x^2 + 2x - 16}{4x^2 - 7}$

Show using limits!  $y = \frac{3}{4}$

If  $x > 0$ ,

$$\begin{aligned} y &= \lim_{x \rightarrow \infty} \frac{3x^2 + 2x - 16}{4x^2 - 7} = \lim_{x \rightarrow \infty} \frac{\frac{3x^2 + 2x - 16}{x^2}}{\frac{4x^2 - 7}{x^2}} = \lim_{x \rightarrow \infty} \frac{\frac{3x^2}{x^2} + \frac{2x}{x^2} - \frac{16}{x^2}}{\frac{4x^2}{x^2} - \frac{7}{x^2}} \\ &= \lim_{x \rightarrow \infty} \frac{3 + \frac{2}{x} - \frac{16}{x^2}}{4 - \frac{7}{x^2}} = \frac{3 + 0 - 0}{4 - 0} = \frac{3}{4} \end{aligned}$$

Likewise, we should show what happens as  $x$  gets very small.

if  $x < 0$ , we get the same thing.

$$\lim_{x \rightarrow -\infty} \frac{3 + \frac{2}{x} - \frac{16}{x^2}}{4 - \frac{7}{x^2}} = \frac{3 + 0 - 0}{4 - 0} = \frac{3}{4}$$

This does not always happen.

**Evaluate the following:**

8.  $\int 5 \cos x \, dx = 5 \sin x + c$

9.

$$\begin{aligned} \int \frac{x^2 - 2}{x^2} \, dx &= \int \left[ \frac{x^2}{x^2} - \frac{2}{x^2} \right] \, dx = \int [1 - 2x^{-2}] \, dx = x - 2 \frac{x^{-2+1}}{-2+1} + c \\ &= x - 2 \frac{x^{-1}}{-1} + c = x + \frac{2}{x} + c \end{aligned}$$

10.

$$\int \frac{x^3}{(x^4 + 3)^4} \, dx$$

$$\text{Let } u = x^4 + 3$$

$$du = 4x^3 \, dx$$

$$\frac{1}{4} du = x^3 \, dx$$

$$\begin{aligned} \int \frac{1}{u^4} \cdot \frac{1}{4} du &= \frac{1}{4} \int u^{-4} \, du = \frac{1}{4} \cdot \frac{u^{-4+1}}{-4+1} + c \\ &= -\frac{1}{12u^3} + c = -\frac{1}{12(x^4 + 3)^3} + c \end{aligned}$$

Evaluate the following:

11.

$$\int_0^2 x\sqrt{x^2+1} dx \quad \begin{array}{l} \text{Let } u = x^2 + 1 \\ du = 2x dx \\ \frac{1}{2} du = x dx \end{array} \quad \begin{array}{l} \text{If } x = 0, u = 0^2 + 1 = 1 \\ x = 2, u = 2^2 + 1 = 5 \end{array}$$

$$\begin{aligned} \frac{1}{2} \int_1^5 \sqrt{u} du &= \frac{1}{2} \int_1^5 u^{1/2} du = \frac{1}{2} \frac{u^{1/2+1}}{1/2+1} \Big|_1^5 = \frac{1}{2} \frac{2}{3} u^{3/2} = \frac{1}{3} (5^{3/2} - 1^{3/2}) \\ &= \frac{1}{3} (5^{3/2} - 1) \end{aligned}$$

12.  $\int_1^3 5 dx = 5x \Big|_1^3 = 5(3) - 5(1) = 15 - 5 = 10$

13.  $\frac{d}{dx} \left[ \int_0^x \sqrt{t^2+1} dt \right] = \sqrt{x^2+1}$

Extra Credit (if time)

Let  $f(x) = x\sqrt{3-x^2}$ . Graph. Find all important information. (min, max, poi, etc...)

First, let's look at the domain.  $[-\sqrt{3}, \sqrt{3}]$

$$3 - x^2 \geq 0 \Rightarrow 3 \geq x^2 \Rightarrow x^2 \leq 3 \Rightarrow \sqrt{x^2} \leq \sqrt{3} \Rightarrow |x| \leq \sqrt{3}$$

$$-\sqrt{3} \leq x \leq \sqrt{3}$$

Critical values:

$$f'(x) = (1)(3-x^2)^{\frac{1}{2}} + x(\frac{1}{2})(3-x^2)^{\frac{1}{2}-1}(-2x)$$

$$= (3-x^2)^{\frac{1}{2}} - x^2(3-x^2)^{-\frac{1}{2}}$$

$$= (3-x^2)^{-\frac{1}{2}} [3-x^2 - x^2] = \frac{3-2x^2}{(3-x^2)^{\frac{1}{2}}}$$

Setting equal to zero,

$$3 - 2x^2 = 0 \Rightarrow 2x^2 = 3 \Rightarrow c = \pm\sqrt{\frac{3}{2}} \approx \pm 1.22$$

$c = \pm\sqrt{3}$  are undefined. In this case, they are endpoints which may be max's or min's.

$$f''(x) = \frac{(3-x^2)^{\frac{1}{2}}(-4x) - (3-2x^2)(\frac{1}{2})(3-x^2)^{\frac{1}{2}-1}(-2x)}{\left[(3-x^2)^{\frac{1}{2}}\right]^2}$$

$$= \frac{x(3-x^2)^{-\frac{1}{2}} \left[ (3-x^2)^1(-4) + (3-2x^2) \right]}{3-x^2} = \frac{x[2x^2-9]}{(3-x^2)^{\frac{3}{2}}}$$

$k = 0$  only.

Since,  $2x^2 - 9 = 0$  means  $x = \pm\sqrt{4.5}$  which are not in the domain.

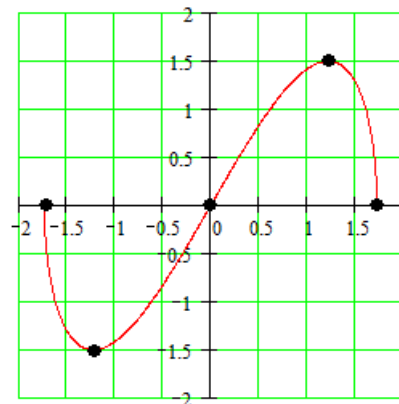
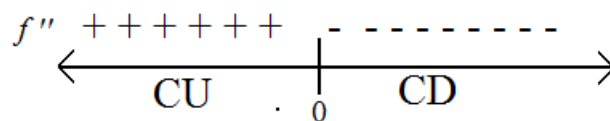
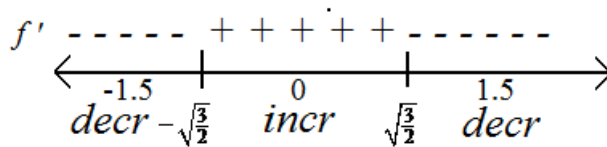
$$f'(-1.5) = \frac{3-2(-1.5)^2}{(3-(-1.5)^2)^{1/2}} \approx -1.7 < 0$$

$$f'(1.5) = \frac{3-2(1.5)^2}{(3-1.5^2)^{1/2}} \approx -1.7 < 0$$

$$f'(0) = \frac{3-2(0)^2}{(3-0^2)^{1/2}} = \sqrt{3} > 0$$

$$f''(-1) = \frac{(-1)[2(-1)^2-9]}{(3-(-1)^2)^{\frac{3}{2}}} = \frac{7}{2^{3/2}} > 0 \therefore CU$$

$$f''(1) = \frac{(1)[2(1)^2-9]}{(3-(1)^2)^{\frac{3}{2}}} = \frac{-7}{2^{3/2}} < 0 \therefore CD$$



Absolute Min:  $(-\sqrt{\frac{3}{2}}, -1.5)$

Rel Min:  $(-\sqrt{3}, 0)$

Absolute Max:  $(\sqrt{\frac{3}{2}}, 1.5)$

Rel Max:  $(-\sqrt{3}, 0)$

POI:  $(0,0)$