Calculus **Test 2** **Name:** ______________________

You must show all work!

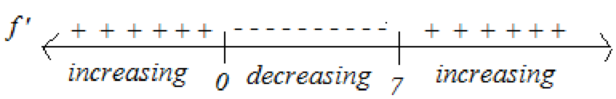
7 points each **Date:** *13 November 2008*

1. $f(x) = 2x^3 - 21x^2 + 5$ Don't Graph!

(a) Find all critical numbers for the function.
(b) Find the intervals where the function is increasing or decreasing.
(c) Apply the first or second derivative to find the extrema.

(a) $f'(x) = 6x^2 - 42x = 6x(x-7) = 0;\ c = 0, 7$

(b) Test points:



$$f\,'(-1) = 6(-1)(-1-7) = -6(-8) = 48 > 0$$
$$f\,'(11) = 6(1)(1-7) = 6(-6) = -36 < 0$$
$$f\,'(8) = 6(8)(8-7) = 6(8) = 48 > 0$$

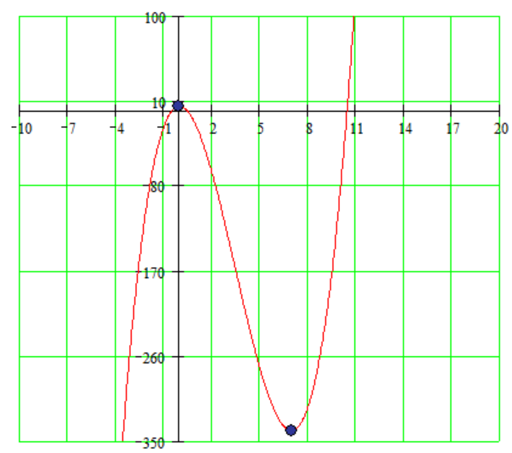
(c) First derivative Test: (FDT)

Since $f\,'(x) > 0$ on $(-\infty, 0)$ and $f\,'(x) < 0$ on $(0, 7)$, by the FDT, (0, 5) is a relative maximum.

Since $f\,'(x) < 0$ on $(0, 7)$ and $f\,'(x) > 0$ on $(0, \infty)$, by the FDT, (7, –338) is a relative minimum.

Remark:

$$f(0) = 0 + 0 + 5 = 5$$
$$f(7) = 2(7^3) - 21(7^2) + 5 = -338$$



You could also use the second derivative test.

$f\,''(x) = 12x - 42$

$f\,''(0) = 12(0) - 42 = -42 < 0,\ \therefore CD, (0, 5)$ is rel. max..

$f\,''(7) = 12(7) - 42 = 42 > 0,\ \therefore CU, (7, -338)$ is rel. min..

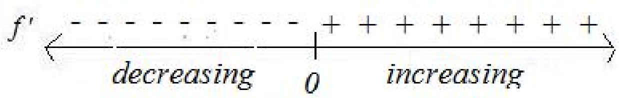
2. $f(x) = \dfrac{x^2}{x^2+4}$ Don't Graph!

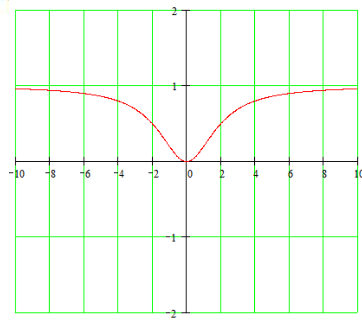
(a) Find all critical numbers for the function.
(b) Find the intervals where the function is increasing or decreasing.
(c) Apply the first or second derivative to find the extrema.

(a) $$f'(x) = \frac{(x^2+4)2x - 2x(x^2)}{\left(x^2+4\right)^2} = \frac{2x^3+8x-2x^3}{\left(x^2+4\right)^2} = \frac{8x}{\left(x^2+4\right)^2}$$

$c = 0$, note: $x^2 + 4 \neq 0$

(b) Test points:





$$f'(-1) = \frac{8(-1)}{(-1)^2+4} = \frac{-8}{5} < 0$$

$$f'(1) = \frac{8(1)}{(1)^2+4} = \frac{8}{5} > 0$$

(c) First derivative Test: (FDT)

Since $f'(x) < 0$ on $(-\infty, 0)$ and $f'(x) > 0$ on $(0, \infty)$, by the FDT, (0, 0) is a relative minimum.

Remark: $f(0) = 0$

You can use the Second Derivative Test but it is harder.

$$f''(x) = \frac{\left(x^2+4\right)^2 8 - 8x(2)\left(x^2+4\right)(2x)}{\left[\left(x^2+4\right)^2\right]^2} = \frac{8\left(x^2+4\right)\left[\left(x^2+4\right) - x(2)(2x)\right]}{\left(x^2+4\right)^4}$$

$$= \frac{8\left(x^2+4\right)\left[x^2+4-4x^2\right]}{\left(x^2+4\right)^4} = \frac{8(4-3x^2)}{\left(x^2+4\right)^3}$$

$$f''(0) = \frac{8(4-3(0^2))}{\left(0^2+4\right)^3} = \frac{32}{64} > 0, \therefore CU. \text{ Hence } (0,0) \text{ is a rel. min..}$$

3. Graph: $g(x) = x^3 - 6x^2 + 6$.

Find all important information. (maxs, mins, poi, etc…) Skip x-intercepts.

$g'(x) = 3x^2 - 12x = 3x(x-4),\ c = 0, 4$

$g''(x) = 6x - 12 = 6(x-2),\ k = 2$

Using 2nd derivative test,

$g''(0) = 6(0) - 12 = -12 < 0,\ \therefore CD,\ (0,6)$ is a rel. max..

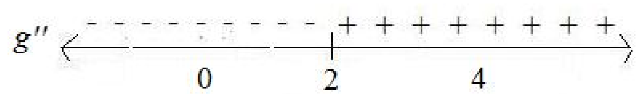
$g''(4) = 6(4) - 12 = 12 > 0,\ \therefore CU,\ (4,-26)$ is a rel. min..

Checking for POI, $k = 2$.

From above,

$g''(0) < 0 \Rightarrow$ Concave Down

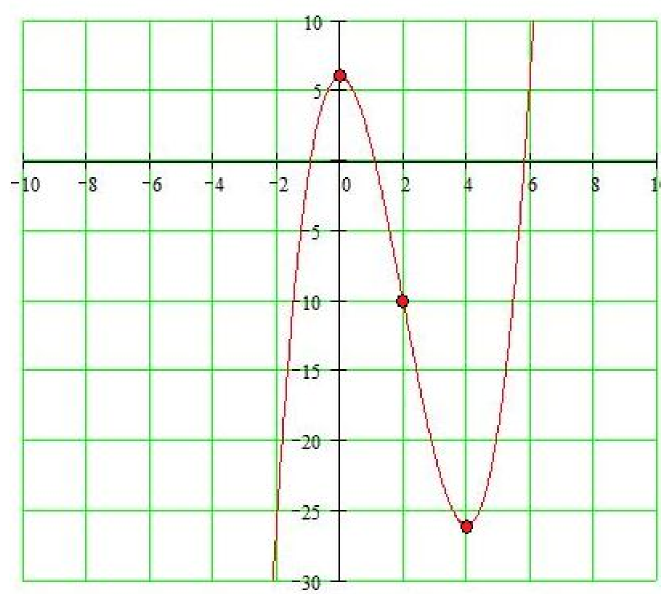
$g''(4) > 0 \Rightarrow$ Concave Up



POI: (2, −10)

Remark: $g(2) = 2^3 - 6(2^2) + 6 = 8 - 24 + 6 = -10$

Of course the y–intercept is (0, 6).



4. **Answer and explain:**

(a) If $f'(3)=0$ and $f''(3)=-5$, then is $f(3)$ is a relative maximum, minimum, p.o.i., or neither.

By the 2nd derivative test, $f(3)$ is a relative maximum because 3 is a critical number and $f''(3) < 0$ means that f is Concave Down.

(b) If $f'(3)=0$ and $f''(3)=0$, then is $f(3)$ is a relative maximum, minimum, p.o.i., or neither.

It is inconclusive if $f(3)$ is a max or min by the second derivative test since $f''(3)=0$ is neither positive or negative. But, since $f''(3)=0$ is a **possible** Point Of Inflection. WE cannot tel for sure unless we know the concavity of the intervals on both sides of 3.

5. $$\lim_{n\to\infty}\sum_{i=1}^{n}\frac{i-3}{n^2} = \lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{n}(i-3) \quad \text{(notice hint)}$$

$$\lim_{n\to\infty}\frac{1}{n^2}\sum_{i=1}^{n}(i-3) = \lim_{n\to\infty}\frac{1}{n^2}\left[\frac{n(n+1)}{2}-3n\right] = \lim_{n\to\infty}\left[\frac{n(n+1)}{2n^2}-\frac{3n}{n^2}\right]$$

$$= \lim_{n\to\infty}\left[\frac{n^2+n}{2n^2}-\frac{3}{n}\right] = \frac{1}{2}-0 = \frac{1}{2}$$

6. Investigate the concavity of the following function. Must show; graphing doesn't count!

$f(x) = \sqrt{x}$

$$f(x) = x^{1/2},\ x \ge 0$$

$$f'(x) = \frac{1}{2}x^{1/2-1} = \frac{1}{2}x^{-1/2} = \frac{1}{2\sqrt{x}};\ x > 0$$

$$f''(x) = -\frac{1}{2}\cdot\frac{1}{2}x^{-1/2-1} = -\frac{1}{4}x^{-3/2} = -\frac{1}{4\sqrt[2]{x^3}} = -\frac{1}{4x\sqrt[2]{x}};\ x \ne 0$$

We only need to test $x > 0$.

$Let\ x = 1,$

$$f''(1) = -\frac{1}{4(1)\sqrt[2]{1}} = -\frac{1}{4} < 0$$

Therefore, f is concave down on the entire domain of $[0, \infty)$.

7. Find the horizontal asymptotes for $f(x) = \dfrac{3x^2 + 2x - 16}{4x^2 - 7}$

Show using limits! $y = \dfrac{3}{4}$

$If\ x > 0,$

$$y = \lim_{x\to\infty} \frac{3x^2 + 2x - 16}{4x^2 - 7} = \lim_{x\to\infty} \frac{\frac{3x^2 + 2x - 16}{x^2}}{\frac{4x^2 - 7}{x^2}} = \lim_{x\to\infty} \frac{\frac{3x^2}{x^2} + \frac{2x}{x^2} - \frac{16}{x^2}}{\frac{4x^2}{x^2} - \frac{7}{x^2}}$$

$$= \lim_{x\to\infty} \frac{3 + \frac{2}{x} - \frac{16}{x^2}}{4 - \frac{7}{x^2}} = \frac{3 + 0 - 0}{4 - 0} = \frac{3}{4}$$

Likewise, we should show what happens as x gets very small.

$if\ x < 0,$ we get the same thing.

$$\lim_{x\to-\infty} \frac{3 + \frac{2}{x} - \frac{16}{x^2}}{4 - \frac{7}{x^2}} = \frac{3 + 0 - 0}{4 - 0} = \frac{3}{4}$$

This does not always happen.

Evaluate the following:

8. $\int 5\cos x\,dx = 5\sin x + c$

9.

$$\int \frac{x^2-2}{x^2}dx = \int\left[\frac{x^2}{x^2}-\frac{2}{x^2}\right]dx = \int\left[1-2x^{-2}\right]dx = x-2\frac{x^{-2+1}}{-2+1}+c$$

$$= x-2\frac{x^{-1}}{-1}+c = x+\frac{2}{x}+c$$

10.

$$\int \frac{x^3}{\left(x^4+3\right)^4}dx$$

$$Let\, u = x^4+3$$

$$du = 4x^3dx$$

$$\frac{1}{4}du = x^3dx$$

$$\int \frac{1}{u^4}\cdot\frac{1}{4}du = \frac{1}{4}\int u^{-4}du = \frac{1}{4}\cdot\frac{u^{-4+1}}{-4+1}+c$$

$$= -\frac{1}{12u^3}+c = -\frac{1}{12(x^4+3)^3}+c$$

Evaluate the following:

11.

$$\int_0^2 x\sqrt{x^2+1}\;dx \qquad \begin{aligned} &Let\, u = x^2+1 \\ &du = 2x\,dx \\ &\frac{1}{2}du = x\,dx \end{aligned} \qquad \begin{aligned} If\, x = 0,\ u &= 0^2+1 = 1 \\ x = 2,\ u &= 2^2+1 = 5 \end{aligned}$$

$$\frac{1}{2}\int_1^5 \sqrt{u}\;du = \frac{1}{2}\int_1^5 u^{1/2}\;du = \frac{1}{2}\frac{u^{1/2+1}}{1/2+1}\bigg|_1^5 = \frac{1}{2}\frac{2}{3}u^{3/2} = \frac{1}{3}\left(5^{3/2}-1^{3/2}\right)$$

$$= \frac{1}{3}\left(5^{3/2}-1\right)$$

12. $\displaystyle\int_1^3 5\,dx = 5x\Big|_1^3 = 5(3)-5(1) = 15-5 = 10$

13. $\displaystyle\frac{d}{dx}\left[\int_0^x \sqrt{t^2+1}\,dt\right] = \sqrt{x^2+1}$

Extra Credit (if time)

Let $f(x) = x\sqrt{3-x^2}$. Graph. Find all important information. (min, max, poi, etc…)

First, let's look at the domain. $\left[-\sqrt{3},\sqrt{3}\right]$

$$3-x^2 \ge 0 \Rightarrow 3 \ge x^2 \Rightarrow x^2 \le 3 \Rightarrow \sqrt{x^2} \le \sqrt{3} \Rightarrow |x| \le \sqrt{3}$$
$$-\sqrt{3} \le x \le \sqrt{3}$$

Critical values:

$$f'(x) = (1)(3-x^2)^{\frac{1}{2}} + x(\tfrac{1}{2})(3-x^2)^{\frac{1}{2}-1}(-2x)$$
$$= (3-x^2)^{\frac{1}{2}} - x^2(3-x^2)^{-\frac{1}{2}}$$
$$= (3-x^2)^{-\frac{1}{2}}\left[3-x^2-x^2\right] = \frac{3-2x^2}{(3-x^2)^{\frac{1}{2}}}$$

Setting equal to zero,

$$3-2x^2 = 0 \Rightarrow 2x^2 = 3 \Rightarrow c = \pm\sqrt{\frac{3}{2}} \approx \pm 1.22$$

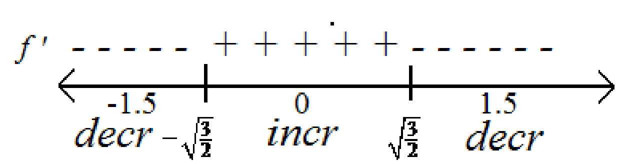
$c = \pm\sqrt{3}$ are undefined. In this case, they are endpoints which may be max's or min's.

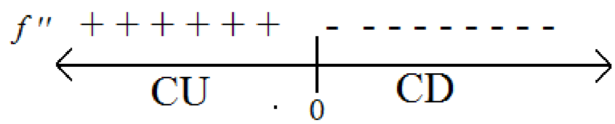
$$f''(x) = \frac{(3-x^2)^{\frac{1}{2}}(-4x) - (3-2x^2)(\frac{1}{2})(3-x^2)^{\frac{1}{2}-1}(-2x)}{\left[(3-x^2)^{\frac{1}{2}}\right]^2}$$
$$= \frac{x(3-x^2)^{-\frac{1}{2}}\left[(3-x^2)^1(-4)+(3-2x^2)\right]}{3-x^2} = \frac{x\left[2x^2-9\right]}{(3-x^2)^{\frac{3}{2}}}$$

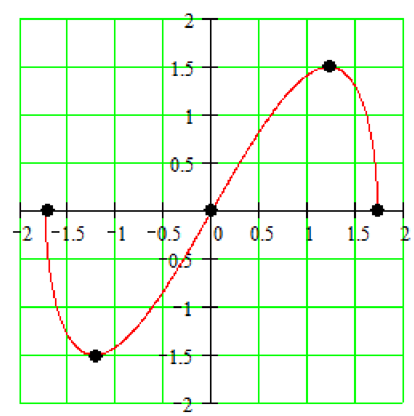
$k = 0$ only.

Since, $2x^2 - 9 = 0$ *means* $x = \pm\sqrt{4.5}$ which are not in the domain.

$$f'(-1.5) = \frac{3-2(-1.5)^2}{(3-(-1.5)^2)^{1/2}} \approx -1.7 < 0$$
$$f'(1.5) = \frac{3-2(1.5)^2}{(3-1.5^2)^{1/2}} \approx -1.7 < 0$$
$$f'(0) = \frac{3-2(0)^2}{(3-0^2)^{1/2}} = \sqrt{3} > 0$$
$$f''(-1) = \frac{(-1)\left[2(-1)^2-9\right]}{(3-(-1)^2)^{\frac{3}{2}}} = \frac{7}{2^{3/2}} > 0 \therefore CU$$
$$f''(1) = \frac{(1)\left[2(1)^2-9\right]}{(3-(1)^2)^{\frac{3}{2}}} = \frac{-7}{2^{3/2}} < 0 \therefore CD$$







Absolute Min: $(-\sqrt{\frac{3}{2}}, -1.5)$

Absolute Max: $(\sqrt{\frac{3}{2}}, 1.5)$

POI: (0,0)

Rel Min: $(-\sqrt{3}, 0)$

Rel Max: $(-\sqrt{3}, 0)$