

You must show all work on this test. No Ti-89 or Ti-92 or ESP.

5 points each

Find the derivative of the following functions.

1. $y = (2x^3 - 7)^{10}$

1. $y' = 60x^2(2x^3 - 7)^9$

$$y' = 10(2x^3 - 7)^{10-1} (2x^3 - 7)' = 10(2x^3 - 7)^{10-1} 6x^2 = 60x^2(2x^3 - 7)^9$$

Power Rule w/Chain Rule

$$y = u^n, \text{ where } u = g(x)$$

$$y' = nu^{n-1} \cdot u'$$

2. $f(x) = \cos(5x)$

2. $f'(x) = -5 \sin(5x)$

$$f'(x) = -\sin(5x)(5) = -5 \sin(5x)$$

Don't forget the chain rule. The argument (or angle) of sine, $5x$, is not affected. You still have to take the derivative of $5x$ which is 5.

3. $h(\theta) = \tan^2(3\theta) = [\tan(3\theta)]^2$

3. $h'(\theta) = 6 \tan(3\theta) \sec^2(3\theta)$

$$h'(\theta) = 2[\tan(3\theta)]^{2-1} \sec^2(3\theta)(3) = 6 \tan(3\theta) \sec^2(3\theta)$$

Note: $\frac{d}{dx} [\tan u] = \sec^2 u \cdot u' = u' \sec^2 u$ The angle does not change when taking the derivative of a trig function.

4. $y = \frac{x}{\sqrt{x^4 + 4}} = \frac{x}{(x^4 + 4)^{\frac{1}{2}}}$

4. $y' = \frac{4 - x^4}{(x^4 + 4)^{\frac{3}{2}}}$

$$y' = \frac{(x^4 + 4)^{\frac{1}{2}}(1) - x(\frac{1}{2})(x^4 + 4)^{-\frac{1}{2}}(4x^3)}{\left[(x^4 + 4)^{\frac{1}{2}}\right]^2} = \frac{(x^4 + 4)^{\frac{1}{2}} - 2x^4(x^4 + 4)^{-\frac{1}{2}}}{(x^4 + 4)^1}$$

$$= \frac{(x^4 + 4)^{-\frac{1}{2}} \left[(x^4 + 4)^1 - 2x^4 \right]}{(x^4 + 4)^1} = \frac{4 - x^4}{(x^4 + 4)^{\frac{3}{2}}}$$

5 points

Find the $\frac{dy}{dx}$ by implicit differentiation.

5. $5x^2 - 4y^3 = 9$

5. $\frac{dy}{dx} = \frac{5x}{6y^2}$

$$\frac{d}{dx}[5x^2 - 4y^3] = \frac{d}{dx}[9]$$

$$10x - 12y^2 \frac{dy}{dx} = 0$$

$$-12y^2 \frac{dy}{dx} = -10x$$

$$\frac{dy}{dx} = \frac{-10x}{-12y^2} = \frac{5x}{6y^2}$$

5 points

Find the $\frac{dy}{dx}$ by implicit differentiation.

6. $x^2y + xy^3 = 2x$

6. $\frac{dy}{dx} = \frac{2 - 2xy - y^3}{x^2 + 3xy^2}$

$$\frac{d}{dx}[x^2y + xy^3] = \frac{d}{dx}[2x]$$

$$2xy + x^2 \frac{dy}{dx} + (1)y^3 + x(3y^2) \frac{dy}{dx} = 2$$

$$x^2 \frac{dy}{dx} + 3xy^2 \frac{dy}{dx} = 2 - 2xy - y^3$$

$$[x^2 + 3xy^2] \frac{dy}{dx} = 2 - 2xy - y^3$$

$$\frac{dy}{dx} = \frac{2 - 2xy - y^3}{x^2 + 3xy^2}$$

10 points

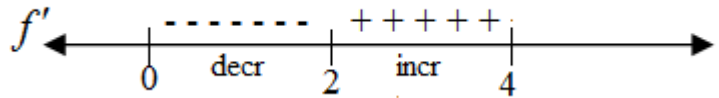
7. Find **all extrema** on the **closed** interval $[0, 4]$ and classify the extrema using the first derivative test.

$$f(x) = x^3 - 12x \qquad 3x^2 - 12 = 3(x^2 - 4) = 0, \quad x^2 - 4 = 0$$

$$f'(x) = 3x^2 - 12 \qquad x^2 = 4 \Rightarrow x = \pm 2$$

The critical values are $c = -2, 2$. Ignore -2 because it is not in $[0, 4]$.

Test points: $f'(1) = 3(1)^2 - 12 = -9 < 0$
 $f'(3) = 3(3)^2 - 12 = 15 > 0$



$$f(0) = 0, \quad f(2) = 2^3 - 12(2) = -16, \quad \text{and} \quad f(4) = 4^3 - 12(4) = 16$$

By the FDT, $(2, -16)$ is a global minimum, $(4, 16)$ is a global maximum, and $(0, 0)$ is a relative maximum.

You may also do this using the definition extrema on a closed interval.

10 points

8. Find **the intervals** where the function is decreasing, increasing or constant.

$$f(x) = x^4 - 4x^3 - 8x^2$$

Find critical points: $f'(x) = 4x^3 - 12x^2 - 16x, \quad 4x(x^2 - 3x - 4) = 4x(x - 4)(x + 1)$
 $x = 0, x - 4 = 0, x + 1 = 0$ gives $c = -1, 0, 4$

interval	$(-\infty, -1)$	$(-1, 0)$	$(0, 4)$	$(4, \infty)$
$f'(x)$	$f'(-2) = -48 < 0$	$f'(-\frac{1}{2}) = \frac{9}{2} > 0$	$f'(-2) = -24 < 0$	$f'(5) = 120 > 0$
conclusion	decreasing	increasing	decreasing	increasing

Or use line graph method.

10 points

9. Let $f''(x) = 4x^3 - 2x$ and let $f(x)$ have critical numbers $-2, 0, 1$. Use the **second derivative** test to determine if any of the critical numbers gives a local minimum or maximum.

$$f''(-2) = 4(-2)^3 - 2(-2) = -8 < 0 \qquad \text{By SDT, (local) maximum. Concave Down}$$

$$f''(0) = 4(0)^3 - 2(0) = 0 \qquad \text{By SDT, no conclusion.}$$

$$f''(1) = 4(1)^3 - 2(1) = 2 > 0 \qquad \text{By SDT, (local) minimum. Concave Up}$$

This is the Second Derivative Test.

10 points

11. Find the open intervals on which the graph is concave up or concave down.

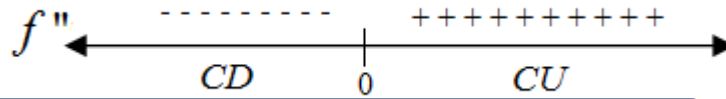
$$f(x) = x^5 - 5x$$

We need to find the second derivative so we can test for concavity.

$$f'(x) = 5x^4 - 5$$

$$f''(x) = 20x^3$$

So possible POI is $k = 0$.



Therefore, f is concave up down on $(-\infty, 0)$ and concave up on $(0, \infty)$.

10 points

12. Find the all extrema for $f(x) = 2x\sqrt{x+3}$ using any method.

$$f'(x) = 2 \left[(1)(x+3)^{\frac{1}{2}} + x \left(\frac{1}{2} \right) (x+3)^{\frac{1}{2}-1} (1) \right] = 2(x+3)^{\frac{1}{2}} + x(x+3)^{-\frac{1}{2}}$$

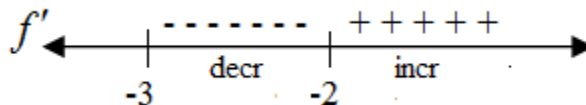
$$= (x+3)^{-\frac{1}{2}} \left[2(x+3)^1 + x \right] = \frac{3x+6}{(x+3)^{\frac{1}{2}}}$$

$$\frac{3x+6}{(x+3)^{\frac{1}{2}}} = 0 \Rightarrow 3x+6 = 0 \Rightarrow x = -2 \quad \text{Note: } f'(-3) = \text{undefined}$$

$$f'(-2.5) = \frac{3(-2.5)+6}{(-2.5+3)^{\frac{1}{2}}} = \frac{-1.5}{\sqrt{0.5}} < 0$$

$c = -2, -3$ are critical numbers.

$$f'(0) = \frac{3(0)+6}{(0+3)^{\frac{1}{2}}} = \frac{6}{\sqrt{3}} > 0$$



$$f(-3) = 2(-3)\sqrt{-3+3} = -6(0) = 0, \quad f(-2) = 2(-2)\sqrt{-2+3} = -4(1) = -4$$

By the FDT, $(-3, 0)$ is a local maximum and $(-2, -4)$ is a minimum (global).

10 points

For #13 $f(x) = 2x^3 - 3x^2$

a) Find all critical numbers of f .

$$f'(x) = 6x^2 - 6x$$
$$6x(x-1) = 0 \quad x = 0 \text{ or } x = 1$$

$c = 0, 1$

b) Locate local extrema using the **second derivative test**.

$f''(x) = 12x - 6$, Therefore, by the SDT

$f''(0) = 12(0) - 6 = -6 < 0 \quad (0, 0) \text{ is a maximum (local)}$
 $f''(1) = 12(1) - 6 = 6 > 0 \quad (1, -1) \text{ is a minimum (local)}$

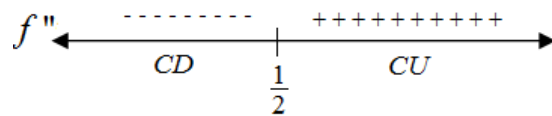
Note: $f(0) = 2(0)^3 - 3(0)^2 = 0$ and $f(1) = 2(1)^3 - 3(1)^2 = -1$

3 Bonus Points

Determine points of inflection using concavity.

$$f''(x) = 12x - 6 = 6(2x - 1) = 0, \quad 2x - 1 = 0 \Rightarrow x = \frac{1}{2}$$

$$k = \frac{1}{2}, \text{ a possible point of inflection.}$$



Therefore, $(\frac{1}{2}, -\frac{1}{2})$ is a Point Of Inflection.

$$f\left(\frac{1}{2}\right) = 2\left(\frac{1}{2}\right)^3 - 3\left(\frac{1}{2}\right)^2 = \frac{2}{8} - \frac{3}{4} = -\frac{1}{2}$$

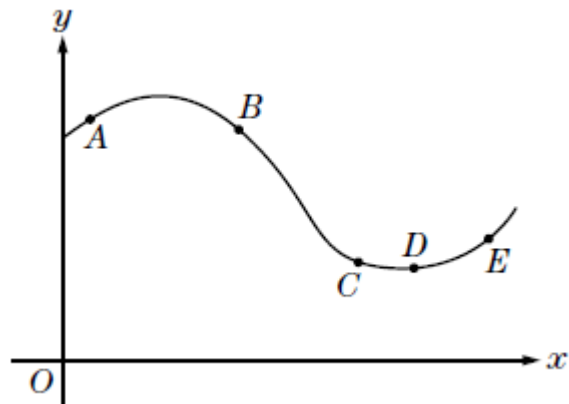
10 points

At which of the five points on the graph in the figure at the right is ...

14. $\frac{dy}{dx} > 0$ Increasing. A & E

15. $\frac{d^2y}{dx^2} < 0$ Concave Down: A & B

2 Bonus points
 $\frac{dy}{dx} = 0$ Max or Min: D



Extra Credit

A spherical balloon is inflated with gas at the rate of 800 cubic centimeters per minute. How fast is the radius of the balloon increasing at the instant the radius is

$$V = \frac{4}{3} \pi r^3$$

$$\frac{dV}{dt} = 4\pi r^2 \frac{dr}{dt}$$

(a) 25 cm ?

$$800 = 4\pi(25)^2 \frac{dr}{dt}$$
$$\frac{dr}{dt} = \frac{800}{4\pi(25)^2} = \frac{8}{25\pi} \text{ cm / min}$$

(b) 50 cm ?

$$800 = 4\pi(50)^2 \frac{dr}{dt}$$
$$\frac{dr}{dt} = \frac{800}{4\pi(50)^2} = \frac{2}{25\pi} \text{ cm / min}$$

You are finished. Yeah you!!!!