Math 182 Test 3 Name _____KEY__________

McDonald Date 4/19/2004

7 Points each

1. Evaluate $\int \sin^3 x \; dx$

Let $u = \cos x$, $du = -\sin x\, dx$

$$\int \sin^3 x \; dx = \int \sin^2 x \cdot \sin x \; dx = \int (1-\cos^2 x)\sin x \; dx = -\int (1-u^2)\, du$$

$$\int (-1+u^2)\, du = -u + \tfrac{1}{3}u^3 + C = \boxed{-\cos x + \tfrac{1}{3}\cos^3 x + C}$$

2. Evaluate $\int_0^{\frac{\pi}{4}} \tan^3 x \; dx$

$$\int \tan^3 x \; dx = \int \tan^2 x \cdot \tan x \; dx = \int (\sec^2 x - 1)\tan x \; dx = \int (\sec^2 x \cdot \tan x - \tan x) \; dx$$

$$= \int (\sec^2 x \cdot \tan x) \; dx - \int \tan x \; dx$$

First Integral

$$\int \sec^2 x \cdot \tan x \; dx = \int u \cdot du = \tfrac{1}{2}u^2 + C_1 = \tfrac{1}{2}\tan^2 x + C_1$$

Let $u = \tan x$, $du = \sec^2 x\, dx$

Second Integral

$$\int \tan x \; dx = -\ln|\cos x| + C_2$$

Put it all together

$$\int_0^{\frac{\pi}{4}} \sec^2 x \tan x dx - \int_0^{\frac{\pi}{4}} \tan x dx = \tfrac{1}{2}\tan^2 x - \left(-\ln|\cos x|\right)\Big]_0^{\frac{\pi}{4}}$$

$$= \tfrac{1}{2}\tan^2 x + \ln|\cos x|\Big]_0^{\frac{\pi}{4}} = \tfrac{1}{2}\tan^2 \tfrac{\pi}{4} + \ln\left|\cos\tfrac{\pi}{4}\right| - \left(\tfrac{1}{2}\tan^2 0 + \ln|\cos 0|\right)$$

$$= \tfrac{1}{2}(1) + \ln\tfrac{\sqrt{2}}{2} - 0 - 0 = \boxed{\tfrac{1}{2} + \ln\tfrac{\sqrt{2}}{2}}$$

$$\frac{1}{2} + \ln\left(\frac{1}{2}\right)^{\frac{1}{2}} = \frac{1}{2} + \frac{1}{2}\ln\left(\frac{1}{2}\right) = \frac{1+\ln\frac{1}{2}}{2}$$

Ti-89 Calculator Answer

3. Evaluate $\int \frac{\sin^2 x}{\cos x}\,dx$

$$\int \frac{1-\cos^2 x}{\cos x}\,dx = \int \frac{1}{\cos x} - \frac{\cos^2 x}{\cos x}\,dx = \int \sec x - \cos x\,dx$$

$$= \ln\left|\sec x + \tan x\right| - \cos x + C$$

4. Evaluate $\int \frac{-x+1}{2x^2+x}\,dx$

Partial Fractions

$$\frac{-x+1}{x(2x+1)} = \frac{A}{x} + \frac{B}{2x+1} \Rightarrow -x+1 = A(2x+1) + Bx$$

If $x = 0$

$$-0+1 = A(2\cdot 0+1) + B(0)$$
$$1 = A$$

If $x = -\frac{1}{2}$

$$-(-\tfrac{1}{2}) + 1 = A\left(2(-\tfrac{1}{2})+1\right) + B(-\tfrac{1}{2})$$
$$\tfrac{3}{2} = -\tfrac{1}{2}B \Rightarrow B = -3$$

$$\int \frac{-x+1}{2x^2+x}\,dx = \int \frac{1}{x} - \frac{3}{2x+1}\,dx = \ln|x| - 3\cdot\tfrac{1}{2}\ln|2x+1| + C = \ln|x| - \tfrac{3}{2}\ln|2x+1| + C$$

REMARK: $\int \frac{3}{2x+1}dx$ Let

$$u = 2x+1$$
$$du = 2dx$$
$$\tfrac{1}{2}du = dx$$

$$\int \frac{3}{2x+1}dx = 3\cdot\tfrac{1}{2}\int \frac{1}{u}du = \tfrac{3}{2}\ln|u| + C = \tfrac{3}{2}\ln|2x+1| + C$$

5. Evaluate $\int \frac{3x^2-6x+9}{(x^2+9)(x-3)}\,dx$

Partial Fractions

$$\frac{3x^2-6x+9}{(x^2+9)(x-3)} = \frac{Ax+B}{x^2+9}+\frac{C}{x-3} \quad \Rightarrow \quad 3x^2-6x+9=(Ax+B)(x-3)+C(x^2+9)$$

$$3x^2-6x+9 = Ax^2-3Ax+Bx-3B+Cx^2+9C=(A+C)x^2+(B-3A)x+(-3B+C)$$

$$A+C=3$$
$$-3A+B=-6$$
$$-3B+9C=9$$

Notice If $x=3$, then $3(3)^2-6(3)+9=(A(3)+B)(3-3)+C(3^2+9)$

$$27-18+9=C(9+9)$$
$$18=18C \quad \Rightarrow \quad \boldsymbol{C=1}$$

If $C=1$, then $A+1=3 \quad \Rightarrow \quad \boldsymbol{A=2}$

If $A=2$, then $-3(2)+B=-6 \quad \Rightarrow \quad \boldsymbol{B=0}$

$$\int \frac{3x^2-6x+9}{(x^2+9)(x-3)}\,dx = \int \frac{2x}{x^2+9}+\frac{1}{x-3}\,dx = \boxed{\ln\left|x^2+9\right|+\ln|x-3|+C}$$

6. Evaluate $\int \frac{1}{x^2-2x+2}dx$

Complete Square

$$\int \frac{1}{x^2-2x+2}dx = \int \frac{1}{x^2-2x+1-1+2}\,dx = \int \frac{1}{(x-1)^2+1}\,dx$$

Let $u=x-1$

$du=dx$

$$\int \frac{1}{u^2+1}\,dx = \frac{1}{1}\tan^{-1}\frac{u}{1}+C = \boxed{\tan^{-1}(x-1)+C}$$

Find the limit if it exists and state the form of indeterminate for each part.

7. $\lim_{x\to\infty} \frac{x^2-1}{e^x+1} = \lim_{x\to\infty} \frac{2x}{e^x} = \lim_{x\to\infty} \frac{2}{e^x} = \boxed{\infty}$

Indeterminate form $\frac{\infty}{\infty}$ $\frac{\infty}{\infty}$

Use L'Hôpital's rule twice

8. $\lim_{x\to 1} \frac{\ln x}{x-1} = \lim_{x\to 1} \frac{\frac{1}{x}}{1} = \lim_{x\to 1} \frac{1}{x} = \frac{1}{1} = \boxed{1}$

Indeterminate form $\frac{0}{0}$

Use L'Hôpital's rule once

9. $\lim_{x\to 0^+} x^x$

Use this method

$y = x^x$

$\ln y = \ln x^x$

$\ln y = x \ln x$

$$\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} x\ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{-x^2}{x} = \lim_{x\to 0^+} -x = -0 = 0$$

Indeterminate form $0\cdot\infty$, but now $\frac{\infty}{\infty}$. Use L'Hôpital's rule once

$$\lim_{x\to 0^+} \ln y = \ln \lim_{x\to 0^+} x^x = 0 \Rightarrow \lim_{x\to 0^+} x^x = e^0 = \boxed{1}$$

Find the limit if it exists and state the form of indeterminate for each part.

10. $\lim\limits_{x\to 0^+} x^2 \ln x$

$$\lim_{x\to 0^+} x^2 \ln x = \lim_{x\to 0^+} \frac{\ln x}{\frac{1}{x^2}} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{-\frac{2}{x^3}} = \lim_{x\to 0^+} \frac{-x^3}{2x} = \lim_{x\to 0^+} -\frac{x^2}{2} = -\frac{0^2}{2} = \boxed{0}$$

Indeterminate form $0 \cdot \infty$, but now $\frac{\infty}{\infty}$. Use L'Hôpital's rule once

11. Evaluate $\int \sin^2 4x \, dx$

Remark $\sin^2 \theta = \dfrac{1-\cos 2\theta}{2}$

Let $\theta = 4x$

$$\sin^2 4x = \frac{1-\cos 2(4x)}{2} = \frac{1-\cos 8x}{2}$$

$$\int \sin^2 4x \, dx = \int \frac{1-\cos 8x}{2} = \frac{1}{2}\int 1-\cos 8x \, dx = \frac{1}{2}\left(x - \frac{1}{8}\sin 8x\right) + C = \boxed{\frac{1}{2}x - \frac{1}{16}\sin 8x + C}$$

12. Evaluate using $\sin x = \dfrac{2u}{1+u^2}$, $\cos x = \dfrac{1-u^2}{1+u^2}$ and $dx = \dfrac{2}{1+u^2}du$ where $u = \tan\dfrac{x}{2}$.

Evaluate $\displaystyle\int \frac{1}{\tan x + \sin x}dx$

$$\tan x = \frac{\sin x}{\cos x} = \frac{\dfrac{2u}{1+u^2}}{\dfrac{1-u^2}{1+u^2}} = \frac{2u}{1+u^2}\cdot\frac{1+u^2}{1-u^2} = \frac{2u}{1-u^2}$$

$$\tan x + \sin x = \frac{2u}{1-u^2} + \frac{2u}{1+u^2} = \frac{2u(1+u^2)+2u(1-u^2)}{(1-u^2)(1+u^2)} = \frac{2u+2u^3+2u-2u^3}{(1-u^2)(1+u^2)} = \frac{4u}{(1-u^2)(1+u^2)}$$

$$\int \frac{1}{\tan x + \sin x}dx = \int \frac{1}{\dfrac{4u}{(1-u^2)(1+u^2)}}\frac{2}{1+u^2}du = \int \frac{(1-u^2)(1+u^2)}{4u}\frac{2}{1+u^2}du = \int \frac{(1-u^2)}{2u}du$$

$$= \int\left[\frac{1}{2u}-\frac{u^2}{2u}\right]du = \int\left[\frac{1}{2u}-\frac{u}{2}\right]du = \tfrac{1}{2}\ln|u| - \tfrac{1}{4}u^2 + C = \boxed{\tfrac{1}{2}\ln\left|\tan\tfrac{x}{2}\right| - \tfrac{1}{4}\tan^2\tfrac{x}{4} + C}$$

13. Evaluate $\displaystyle\int \frac{1}{\sqrt[4]{x}+\sqrt[3]{x}}dx$

Let

$u = x^{\frac{1}{12}}$

$u^{12} = x$

$12u^{11} = dx$

$$\int \frac{1}{\sqrt[4]{u^{12}}+\sqrt[3]{u^{12}}}\cdot 12u^{11}du = 12\int \frac{u^{11}}{u^3+u^4}\cdot du = 12\int \frac{u^8}{1+u}\cdot du =$$

Using Long Division…

$$\frac{u^8}{1+u} = u^7 - u^6 + u^5 - u^4 + u^3 - u^2 + u - \frac{1}{1+u}$$

$$12\int \frac{u^8}{1+u}\cdot du = 12\int\left[u^7 - u^6 + u^5 - u^4 + u^3 - u^2 + u - \frac{1}{1+u}\right]\cdot dx$$

$$= 12\left(\tfrac{1}{8}u^8 - \tfrac{1}{7}u^7 + \tfrac{1}{6}u^6 - \tfrac{1}{5}u^5 + \tfrac{1}{4}u^4 - \tfrac{1}{3}u^3 + \tfrac{1}{2}u^2 - \ln|1+u|\right) + C$$

$$\boxed{= \tfrac{4}{3}x^{\frac{3}{4}} - \tfrac{12}{7}x^{\frac{7}{12}} + 2x^{\frac{1}{2}} - \tfrac{12}{5}x^{\frac{5}{12}} + 3x^{\frac{1}{3}} - 4x^{\frac{1}{4}} + 6x^{\frac{1}{6}} - 12\ln\left|1+x^{\frac{1}{12}}\right| + C}$$

14. Use this silly formula $\int \sin au \cos bu \; du = -\frac{\cos(a-b)u}{2(a-b)} - \frac{\cos(a+b)u}{2(a+b)} + C$

Evaluate $\int \sin 5x \cos 4x \; dx$ Here $a = 5$ and $b = 4$

$$\int \sin 5x \cos 4x \; du = -\frac{\cos(5-4)x}{2(5-4)} - \frac{\cos(5+4)x}{2(5+4)} + C = \boxed{-\frac{\cos x}{2} - \frac{\cos 9x}{18} + C}$$

Extra Credit: No Stinking Calculators Either!!! PICK ONE. ONLY ONE!!!

Evaluate $\int \frac{1}{x(\sqrt{x}+\sqrt[4]{x})} dx$

Let $u = x^{\frac{1}{4}}$, $u^4 = x$, $4u^3 du = dx$

$$\int \frac{4u^3 du}{u^4\left(\sqrt{u^4}+\sqrt[4]{u^4}\right)} = 4\int \frac{u^3 du}{u^4\left(u^2+u\right)} = 4\int \frac{du}{u\left(u^2+u\right)} = 4\int \frac{du}{u^2(u+1)}$$

Partial Fraction

$$4\int \frac{1}{u^2(u+1)} du = 4\int \left[-\frac{1}{u}+\frac{1}{u^2}+\frac{1}{u+1}\right] du = \boxed{4\left(-\ln\left|x^{\frac{1}{4}}\right| - \frac{1}{x^{\frac{1}{4}}} + \ln\left|x^{\frac{1}{4}}+1\right|\right) + C = 4\left(-x^{-\frac{1}{4}} + \ln\left|\frac{x^{\frac{1}{4}}+1}{x^{\frac{1}{4}}}\right|\right) + C}$$

Evaluate $\int x\sqrt{5-3x}\; dx$

Let $u = 5-3x$, $\frac{u-5}{-3} = x$, $-\frac{1}{3} du = dx$

$$\int \frac{u-5}{-3}\sqrt{u}\left(-\frac{1}{3}\right) du = \frac{1}{9}\int (u-5)u^{\frac{1}{2}} du = \frac{1}{9}\int (u^{\frac{3}{2}} - 5u^{\frac{1}{2}}) du$$

$$= \frac{1}{9}\cdot\frac{2}{5}u^{\frac{2}{5}} - \frac{5}{9}\cdot\frac{2}{3}u^{\frac{2}{3}} + C = \boxed{\frac{2}{45}(5-3x)^{\frac{2}{5}} - \frac{10}{27}(5-3x)^{\frac{2}{3}} + C}$$